Mathematics and Faith

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In this reflection about mathematics I shall confine myself to arithmetic, the study of the numbers 0, 1, 2, 3, 4, ... Everyone has at least the feeling of familiarity with arithmetic, and the issues that concern the human search for truth in mathematics are already present in arithmetic.

Here is an illustration of research in arithmetic. About 2500 years ago, the Pythagoreans defined a number to be *perfect* in case it is the sum of all its divisors other than itself. Thus $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are perfect. The Pythagoreans, or perhaps Euclid himself, proved that if $2^n - 1$ is a prime, then $(2^n - 1) \cdot 2^{n-1}$ is perfect.[1] More than 2000 years later, Euler proved that every even perfect number is of this form. This left open the question whether there exists an odd perfect number. The search for an odd perfect number or, alternatively, for a proof that no odd perfect number exists, continues today, several centuries after Euler and in the fourth millennium from Pythagoras.

No other field of human endeavor so transcends the barriers of time and culture. What accounts for the astounding ability of Pythagoras, Euler, and mathematicians of the 21st century to engage in a common pursuit?

Of the three schools of thought on the foundations of mathematics—Platonic realism, intuitionism, and formalism—the Platonists offer what seems to be the simplest explanation. The sequence of numbers is the most primitive mathematical structure. Mathematicians postulate as axioms certain self-evident truths about the numbers and then deduce by logical reasoning other truths about numbers, theorems such as those of Euclid and Euler. This is the traditional story told about mathematics. Can there be any philosophical—let alone theological—problem with it? Let us look more closely.

Reasoning in mathematics. The logic of Aristotle—the greatest logician before Gödel—is inadequate for mathematics. It was already inadequate for the mathematics of his day. Only relatively recently was the logic of mathematical reasoning clarified. Boole[2] brilliantly began the clarification and Hilbert[3] perfected it. They succeeded where Frege and Russell went partly astray, because they sharply distinguished syntax from semantics.

How is the syntax of arithmetic formulated? One begins with a few marks whose only relevant property is that they can be distinguished from each other. An *expression* is any combination of these marks written one after the other. Certain expressions, built up according to simple definite rules, are *formulas*, generically denoted by F. Certain formulas are chosen as *axioms*. With a suitable choice of axioms, we obtain what is called *Peano Arithmetic* (PA).

A *proof* in PA is a succession of formulas such that each of its formulas F is either an axiom or is preceded by some formula F' and by $F' \to F$. (One of the marks is $\to$.)[4] Each formula in the proof is a *theorem*.

And this is what mathematics is. Mathematicians are people who prove theorems; we construct proofs.

The salient feature of syntax is that it is concrete. The question whether a putative proof is indeed a proof is a matter simply of checking. Disputes about the correctness of a proof are quickly settled and the mathematical community reaches permanent consensus. The status, age, and reputation of the parties to the dispute play no role. In this we are singularly blessed. (Of course, being human, we do squabble—not about the correctness of a proof but about priority and value. The two can conflict. Once at a party a friend spoke eloquently and at length. He was irate at the credit accorded to another's work. When he concluded, I said with some trepidation, "You seem to be saying two things: the work is utterly trivial, and you did it first." I was relieved when my friend replied, "That is *precisely* what I am saying.")

Admittedly, I have somewhat exaggerated the mechanical nature of proof checking as it occurs in the year 2000, but we can expect it to be fully mechanized, by computer programs, in the near future. To check a proof is mechanical, to construct a proof is not. Living mathematics is one of the glories of the human spirit, and the sources of mathematical creativity are deep and mysterious. In December 1996 it was announced that a computer had solved the Robbins problem. This is a problem that the eminent mathematician Alfred Tarski, and others, had failed to solve. I was interested in this, and having reached what Irving Kaplansky calls the age of ossification when the only way to learn something new is to teach it, I gave a graduate course on this work. What impressed me was not the speed and power of the computer, but the great ingenuity with which William McCune[5] had developed a program for this general kind of problem, and his artful strategy that steered the search between the Scylla of exponentially expanding generation of useless formulas and the Charybdis of premature termination of the search. Computers will play an increasing role in the construction of proofs, but I predict that 2500 years from now they will not have replaced mathematicians.

Hilbert's logic expresses something very deep within human nature. The work of the best and clearest mathematicians of antiquity—Archimedes is the prime example—in no way sounds alien to our ears. Archimedes reasoned in mathematics just as we reason today.

In describing mathematical syntax, I have implicitly advanced an answer to the question raised in the beginning: we can continue the work of Euclid and Euler because we reason in the same way. One might object that this answer is inadequate because it makes no mention of truth.

Truth in mathematics. Now let us turn to the semantics of mathematics, and of arithmetic in particular. I shall describe Platonic semantics.

Among the marks are 0 and S. A *numeral* is an expression of the form 0, S0, SS0, and so forth, generically denoted by n. The mark 0 denotes the number zero, and if the numeral n denotes a certain number then Sn denotes its successor, the next number after it. Thus the numerals serve as names for the numbers.

Certain expressions are *variables*, generically denoted by v. Occurrences of variables in formulas are syntactically divided into *free* and *bound* occurrences, and a formula with no free occurrences of variables is *closed.* The result of substituting the numeral n for each free occurrence of the variable v in the formula F is denoted by $F_v[n]$.

One assigns to each closed formula a *truth value*, true or false. This proceeds iteratively, starting with the simplest formulas and working up to more complex formulas. One of the marks is $\forall$, and one step in the iterative assignment of truth values to formulas is the clause:

$\forall vF$ is true in case for all numerals n, the formula $F_v[n]$ is true.

Platonists maintain that by virtue of the iterative assignment of truth values, a closed formula acquires meaning as a statement about numbers and is either true or false.

The clause displayed above is the battlefield where clash the armies of Platonists, intuitionists, and formalists. It differs from syntactical definitions because it invokes the notion of an infinite search. We have left the realm of the concrete for the speculative.

The intuitionist, epistemologically inclined, says, "What you said makes no sense. How can I possibly examine for every numeral n the truth of $F_v[n]$?" The Platonist, ontologically inclined, replies, "Nevertheless, either $F_v[n]$ is true for every numeral n or there exists a numeral n for which it is false." The formalist is content to observe that the definition of truth value can be formalized in set theory (a branch of mathematics more complex than arithmetic), thus sweeping the semantic dirt under the carpet of syntax.

Intuitionism was the creation of Brouwer, who proposed it as a restricted, and truthful, replacement for classical mathematics. He developed a new syntax and a new semantics, which for arithmetic were clarified and made precise by Heyting and Kleene, respectively. There is a specific closed formula of arithmetic that is false for Platonists and true for intuitionists.[6] When speaking of truth in mathematics, one must always specify truth according to whom.

The notion of truth in mathematics is irrelevant to what mathematicians do, it is vague unless abstractly formalized, and it varies according to philosophical opinion. In short, it is formal abstraction masquerading as reality.

At this point I pause to say something with all the emphasis I can muster. In these days when postmodernism is still in vogue and some people seriously proclaim that scientific truth is a social construct, I do not want to be misunderstood. I have expressed doubts about the coherence of the notion of truth in mathematics, but I am speaking only about *mathematics*.

However much amplification the following description of truth may require, truth is a correspondence between a linguistic formulation and reality. My claim is that there is no Platonic reality underlying mathematics; mathematicians prove theorems, but the theorems are not *about* anything. This is how mathematics differs profoundly from science.

Mathematicians no more *discover* theorems than the sculptor discovers the sculpture inside the stone. (Surely you are joking, Mr. Buonarroti!) But unlike sculpting, our work is tightly constrained, both by the strict requirements of syntax and by the collegial nature of the enterprise. This is how mathematics differs profoundly from art.

To deny the cogency of the Platonic notion of truth in mathematics in no way deprives mathematics of meaning. In mathematics, meaning is found not in a cold, abstract, static world of Platonic ideas but in the human, historical, collegial world of mathematicians and their work. Since Boole, mathematics is understood as the creation and study of abstract patterns and structures; this replaces the old understanding of mathematics as the study of number and magnitude. Meaning is found in the beauty and depth of these patterns, in unsuspected relationships between structures that previously seemed unrelated, in the fierce struggle to bring order to seemingly insurmountable complexity, in the joy of providing new tools for the better understanding of the physical world, in the pleasures of collegial cooperation and competition, in the visual beauty of geometry and dynamics in three dimensions and their inner visual beauty in higher dimensions, and above all in the awe of confronting the potential infinity that is the world of mathematics.

It is sometimes claimed that mathematicians have direct insight that goes beyond proof. A young mathematician would be ill advised to attempt to make a career by advancing such claims; it would violate collegiality. Share with us how you come by your insights so that we can examine your methods, test them, formalize them, see how they fit into the rest of mathematics. If your conjecture is deep and beautiful, if it resists proof or disproof for a long time, you will have made an important contribution to the collegial enterprise—but the final criterion is proof. Proof is the essence of mathematics.

Many, many writers have stated that mathematics is the most certain human acquisition of knowledge. This misperception leads to such embarrassments as the pseudo-Euclidean form that Spinoza gave to his *Ethics*. These writers are too pedestrian in their view of mathematics and yet they give us too much credit. All we do is construct proofs, and at present we do so from axiom systems for which no one can give a convincing demonstration of consistency.

Consistency in mathematics. Unlike truth, consistency is a syntactical matter. One of the marks is $\neg$, and the system is consistent if there do not exist proofs of both a formula F and its *negation* $\neg F$.

Hilbert, stung by the attacks of Brouwer, initiated his program: to establish the consistency of classical mathematics by finitary methods, methods that would be acceptable to the intuitionists. Hilbert called himself a formalist but I suspect that he was a crypto-Platonist. Referring to Cantor's set theory, he declared, "No one shall expel us from the paradise that Cantor has created for us." (It is curious that mathematicians confronting the depths of our subject often fall into the language of faith. Kronecker wrote, "God made the numbers, all else is the work of Man." Once when I expressed my skepticism about the consistency of mathematics to my thesis adviser, the late Irving Segal, he asked, "But you *believe in* the natural numbers, don't you?")

Hilbert's hopes were dashed by Gödel's great paper[7] of 1931. We have seen that mathematical syntax is a matter of combining marks in simple ways; Gödel's first step was to express these combinations within arithmetic. It is frequently said that Gödel's theorem is based on the liar paradox of antiquity ("this statement is not true"), but this misses the point. Gödel replaced the superficial semantic paradox by deep syntax, constructing a formula F of PA that expresses "F is unprovable in PA". This led to his two incompleteness theorems. They are often paraphrased like this:

(1) there is a true but unprovable formula of PA;

(2) if PA is consistent, the formula expressing "PA is consistent" is unprovable in PA.

It is instructive to compare these two statements. The first is a statement about the Platonic world, for as we have seen, the notion of truth for a formula of arithmetic is an extra-mathematical point of contention among the various schools of foundations (unless it is formalized in set theory). The second is a statement about the world we live in, and it demolished Hilbert's program. The first is abstract, the second concrete: Gödel explicitly shows how to derive a contradiction in PA from a proof in PA of the formula expressing "PA is consistent".

The consistency of PA cannot be concretely demonstrated. What are the options? First, Platonism—the option followed by Gödel himself. Second, intuitionism—but this is no solution, since Gödel in a very short paper[8] in 1933 gave an interpretation of classical arithmetic within intuitionistic arithmetic, showing that one is consistent if and only if the other is. Third, pragmatism—to stop worrying about foundations and get on with mathematics; this has been the general response. Fourth, radical formalism—to explore whether PA might not after all be inconsistent.[9]

Abstract ideas have concrete consequences—this is their power, and in human affairs this is their terror, as in the abstract idea of the Aryan race. The Platonist argues that the axioms of PA are true,[10] the rules of inference preserve truth, so all theorems are true, but a formula and its negation cannot both be true; hence PA is consistent. An attempt to construct a contradiction in PA is a concrete act which the Platonist dismisses as futile. For the Platonist the consistency of PA is a matter of faith.

Faith in mathematics. The title of this section should give us pause.

Our culture instills respect for other religions and our faith requires respect for people who practice other religions; nevertheless, I was dismayed when I saw an idol—an actual metal effigy—being worshiped. We who are children of Abraham by adoption may have something to learn from our Jewish and Muslim siblings concerning the vehemence of God's abhorrence of idolatry. God demands our faithful faith.

Christianity in its early years confronted in Hellenistic culture not only pagan idolatry but a refined strain of religious thought in philosophy. In *Fides et Ratio* we read, concerning Greek philosophy, "Superstitions were recognized for what they were and religion was, at least in part, purified by rational analysis."[11] Let us pay close attention to the phrase "at least in part".

Mathematics as we know it originated with Pythagoras, and Pythagoras founded a religion in which numbers played a central role. For the Pythagoreans, the numbers were an infinite, real, uncreated world of beings, and most mathematicians retain that faith today. With no training in history, philosophy, theology, Greek, or Latin, I can only listen as others speak of the early encounter of Christianity with Pythagoreanism. Elaine Pagels writes movingly of Justin's conversion to Christianity after a previous unsatisfactory experience with a Pythagorean teacher who required him first to learn mathematics before he could be enlightened.[12] Perhaps the Holy Spirit had implanted in Justin what today is called "math anxiety".

I must relate how I lost my faith in Pythagorean numbers. One morning at the 1976 Summer Meeting of the American Mathematical Society in Toronto, I woke early. As I lay meditating about numbers, I felt the momentary overwhelming presence of one who convicted me of arrogance for my belief in the real existence of an infinite world of numbers, leaving me like an infant in a crib reduced to counting on my fingers. Now I live in a world in which there are no numbers save those that human beings on occasion construct.

The Pythagorean religion is no longer practiced. But Pythagoras strongly influenced Plato, and through Plato us. The numbers of Pythagoras are the type of Platonic ideas. What are Platonic ideas? Are they created—contingent and subject to change? Are they uncreated—as they were in the beginning, are now, and ever shall be? Or are they constructs of human thought which we come perilously close to idolizing? For us, men and women from the world of learning, to reify abstract ideas, even to base morality on them, is a temptation more insidious than the worship of metal effigies, and more corrupting.

During my first stay in Rome I used to play chess with Ernesto Buonaiuti. In his writings and in his life, Buonaiuti with passionate eloquence opposed the reification of human abstractions. I close by quoting one sentence from his *Pellegrino di Roma*.[13]

"For [St. Paul] abstract truth, absolute laws, do not exist, because all of our thinking is subordinated to the construction of this holy temple of the Spirit, whose manifestations are not abstract ideas, but fruits of goodness, of peace, of charity and forgiveness."

Notes and References

[1] This is Proposition 36, Book IX of Euclid's *Elements*, translated with commentary by Thomas L. Heath, 2nd. ed. revised with additions, Vol. II, Dover, New York, 1956.

[2] George Boole published *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities* in 1854. Bertrand Russell said, "Pure Mathematics was discovered by Boole in a work which he called *The Laws of Thought*," quoted by Carl B. Boyer in *A History of Mathematics*, Wiley, New York, 1968.

[3] David Hilbert and Wilhelm Ackermann, *Gründzuge der theoretischen Logik*, Springer, Berlin, 1928.

[4] It is possible to choose the logical axioms so that the only rule of inference is modus ponens, as I have assumed here for simplicity of exposition. For an exceedingly lucid account of mathematical logic, see Joseph R. Shoenfield, *Mathematical Logic*, Addison-Wesley, Reading, Massachusetts, 1967.

[5] William McCune, *Solution of the Robbins Problem*, Journal of Automated Reasoning, vol. 19, 263–276, 1997. *Robbins Algebras Are Boolean*, online at `http://www-unix.mcs.anl.gov/~mccune/papers/robbins/`

[6] See Stephen Cole Kleene, *Introduction to Metamathematics*, North-Holland, New York, 1952, for a detailed account of a version of Heyting's intuitionistic arithmetic and for Kleene's intuitionistic semantics via recursive realizability. The formula demonstrating the divergence between classical and intuitionistic arithmetic is described in Sec. 82 and is the last corollary in the book. Brouwer's first paper on intuitionism was published, in Dutch, in 1908 (see Kleene's bibliography).

[7] Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik, vol. 38, 173-198, 1931.

[8] Kurt Gödel, *Zum intuitionistischen Arithmetik und Zahlentheorie*, Ergebnisse eines math. Koll., Heft 4 (for 1931-2), 34-38, 1933.

[9] See Edward Nelson, *Predicative Arithmetic*, Mathematical Notes 32, Princeton University Press, Princeton, New Jersey, 1986, for a detailed critique of Peano Arithmetic.

[10] More precisely, the claim is that they are valid; see Shoenfield's book cited above.

[11] John Paul II, *Fides et Ratio*. The quotation is from Article 36. Online at `http://www.vatican.va/holy_father/john_paul_ii/encyclicals/`

[12] Elaine Pagels, *The Origin of Satan*, Random House, New York, 1995. The story of Justin's conversion is told in Chap. 5.

[13] Ernesto Buonaiuti, *Pellegrino di Roma: La generazione dell'esodo*, Laterza, Bari, 1964. The quotation is from the chapter on the years 1915-1920.